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Initial-Boundary Value Problems for an Extensible Beam

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1. INTRODUCTION

In this paper we discuss certain initial-boundary value problems for the nonlinear beam equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left[\beta + k \int_0^l u_\xi(\xi, t)^2 \, d\xi\right] \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where the constants α and k are positive.

Equation (1.1) was proposed by Woinowsky-Krieger [28] as a model for the transverse deflection $u(x, t)$ of an extensible beam of natural length l whose ends are held a fixed distance apart. The nonlinear term represents the change in the tension of the beam due to its extensibility. The model has also been discussed by Eisley [13], while related experimental results have been given by Burgreen [6].

Dickey [10] recently considered the initial-boundary value problem for (1.1) in the case when the ends of the beam are hinged, so that

$$u(0, t) = u(l, t) = u_{xx}(0, t) = u_{xx}(l, t) = 0. \tag{1.2}$$

The initial deflection $u_0(x)$ and the initial velocity $u_1(x)$ of each point x of the beam are assumed given. Dickey showed how the model affords a description of the phenomenon of "dynamic buckling." Assuming a Galerkin expansion for the deflection at time t, he was then able to prove, using a compactness argument, that the resulting infinite system of ordinary differential equations has a unique solution for all time. Dickey has also studied [11] the system of ordinary differential equations corresponding to the case $\alpha = 0$. Equation (1.1) then represents a vibrating string and for certain problems of this kind exact solutions are known (Oplinger [22]).

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The present paper extends the work of Dickey in several directions. We deal with both the case of hinged ends and that of clamped (or built-in) ends for which

$$u(0, t) = u(l, t) = u_x(0, t) = u_x(l, t) = 0. \tag{1.3}$$

In both cases we use the techniques of Lions [19] to prove the existence of weak solutions to the initial–boundary value problem for (1.1). We then show that these solutions satisfy an energy equation and depend continuously on the initial data in a way which implies that the solution for given initial data is unique. The Galerkin method used converges to the solution for an arbitrary basis of the appropriate function spaces. We next prove that when the initial data is sufficiently smooth and satisfies appropriate compatibility conditions, the resulting solution is a classical solution of (1.1). In the hinged-end case the compatibility conditions are linear, but in the clamped-end case they are nonlinear and this makes the regularity proof less straightforward. Our methods also apply to the mixed problem of a beam with one clamped and one hinged end, but for brevity we do not discuss this case.

It would be of interest to extend the analysis of this paper to a more satisfactory model. In a series of papers, Antman [1–3] has used the direct method of the calculus of variations to prove the existence of stable equilibrium configurations for rods and shells with a Cosserat structure. The models used by Antman incorporate both geometric nonlinearities, due to large deflections, and the effects of nonlinear stress–strain laws. He obtains qualitative results on the nature of buckled states. Convexity assumptions analogous to those of Coleman and Noll (see [26]) are essential for the existence proofs. In a similar way we are able to use a monotonicity property (Lemma 6) to establish the convergence of the nonlinear term in (1.1).

The author would like to express his gratitude to Professor Antman for suggesting the present problem and the method of approach used.

In a better model, excluding the effects of damping or fading memory, it is doubtful whether similar regularity properties to those proved here would hold. Zabusky [29], Lax [18], and MacCamy and Mizel [21] have shown that in the special case of one-dimensional motion of a rod, for all nonzero initial conditions breakdown of the solution occurs after a finite time. The breakdown effect disappears when a fading memory assumption is introduced (Greenberg, MacCamy, and Mizel [16]). Our assumption of transverse motion (and thus of uniform tension) may exert a similar smoothing effect.

The effect of adding a linear damping term to (1.1) has been discussed in a recent paper of Reiss and Matkowsky [23], who use a formal asymptotic expansion method to study the approach of the beam to a buckled state.

2. Preliminaries

We first of all explain some notation and introduce some well known function spaces.

Let Ω be the open interval $]0, l[$ of $\mathbb{R}^1$, where $l > 0$ is the length of the beam in its unstressed state. Write $Q = \Omega \times]0, T[$, where $T > 0$ is fixed.

Let $C^m(\Omega)$ be the class of m times continuously differentiable real valued functions on Ω, and set

$$C^\infty(\Omega) = \bigcap_{m=1}^{\infty} C^m(\Omega).$$

Let $\mathscr{D}(\Omega)$ be the subset of $C^\infty(\Omega)$ consisting of those functions with compact support in Ω. $\mathscr{D}(\Omega)$ is given the strict inductive limit topology of L. Schwartz (see Carroll [7]). The dual space of $\mathscr{D}(\Omega)$ is denoted by $\mathscr{D}'(\Omega)$.

In the usual way let $L^2(\Omega)$ be the Hilbert space of real valued Lebesgue measurable functions $f = f(x)$ on Ω with $|f| < \infty$, where

$$|f| = \|f\|_{L^2(\Omega)} = \left(\int_0^l (f(x))^2\, dx\right)^{1/2}.$$

The inner product of two functions f and g in $L^2(\Omega)$ is written

$$(f, g) = \int_0^l f(x)\, g(x)\, dx.$$

We denote by $L^\infty(0, T)$ the class of essentially bounded measurable real valued functions on $]0, T[$. $L^\infty(0, T)$ is a Banach space with the norm

$$\|f\|_{L^\infty(0,T)} = \operatorname*{ess\,sup}_{t \in]0,T[} |f(t)|.$$

The spaces $L^2(Q)$, $C^m(Q)$, $\mathscr{D}(]0, T[)$ and $\mathscr{D}'(]0, T[)$ are defined in the obvious way.

If $g \in C^m(\Omega)$, let

$$\|g\|_m = \left[\sum_{k=0}^{m} \int_0^l \left|\frac{\partial^k g(x)}{\partial x^k}\right|^2 dx\right]^{1/2}.$$

Let $\tilde{C}^m(\Omega)$ be the subset of $C^m(\Omega)$ consisting of those functions g with $\|g\|_m < \infty$. We define the Sobolev space $H^m(\Omega)$ to be the completion of $\tilde{C}^m(\Omega)$ under the norm $\|\ \|_m$.

$H^m(\Omega)$ consists of all functions $u \in L^2(\Omega)$ with strong derivatives $\partial^k u/\partial x^k \in L^2(\Omega)$ for $0 \leqslant k \leqslant m$. The closure of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$ is written

$H_0^m(\Omega)$. $H^m(\Omega)$, $H_0^m(\Omega)$ are both Hilbert spaces. Denote by $H^{-m}(\Omega)$ the dual space of $H_0^m(\Omega)$. We identify $L^2(\Omega)$ with its dual, and hence

$$\mathscr{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) \subset H^{-m}(\Omega) \subset \mathscr{D}'(\Omega).$$

The Hilbert space $H^m(Q)$ is similarly defined, the norm of one of its elements f being

$$\|f\|_{H^m(Q)} = \left[\sum_{r+s\leqslant m} \int_Q \left| \frac{\partial^{r+s} f(x,t)}{\partial x^r \, \partial t^s} \right|^2 dx\, dt\right]^{1/2},$$

where the indices r and s are nonnegative integers and the derivatives are strong derivatives. For general information on Sobolev spaces see Friedman [15].

Let X be a Banach space and let $1 \leqslant p \leqslant \infty$. We say that f belongs to $L^p(0, T; X)$ if f is measurable in t with values in X and is such that

$$\|f\|_{L^p(0,T;X)} < \infty,$$

where

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \, dt\right)^{1/p} \qquad \text{if} \qquad 1 \leqslant p < \infty$$

$$= \operatorname*{ess\,sup}_{t\in]0,T[} \|f(t)\|_X \qquad \text{if} \qquad p = \infty.$$

$L^p(0, T; X)$ is a Banach space (Bourbaki [5]).

We write $\mathscr{D}'(0, T; X) = \mathscr{L}(\mathscr{D}(]0, T[); X)$, the space of continuous linear maps from $\mathscr{D}(]0, T[)$ to X. If $f \in \mathscr{D}'(0, T; X)$, we define $\partial f/\partial t \in \mathscr{D}'(0, T; X)$ by

$$\frac{\partial f}{\partial t}(\varphi) = -f\left(\frac{d\varphi}{dt}\right) \qquad \text{for all } \varphi \in \mathscr{D}(]0, T[). \tag{2.1}$$

$L^p(0, T; X)$ can be embedded 1-1 into $\mathscr{D}'(0, T; X)$. If $f \in L^p(0, T; X)$, define for $\varphi \in \mathscr{D}(]0, T[)$

$$f(\varphi) = \int_0^T f(t)\, \varphi(t)\, dt. \tag{2.2}$$

The integral in (2.2) is a Bochner integral in the Banach space X (cf. Hille and Phillips [17, Chapter III]). By means of (2.2), f may be regarded as belonging to $\mathscr{D}'(0, T; X)$ and may be differentiated with respect to t using (2.1).

For brevity in notation, from now on dots above symbols representing functions denote differentiation with respect to time t, while derivatives with respect to distance x along the beam are written $\partial^m u/\partial x^m = u^{(m)}$. Constants are frequently denoted by C, their dependence on relevant parameters being mentioned where necessary.

3. The Model

Consider an extensible beam whose ends are held at $x = 0$ and $x = l + \Delta$. Let H be the axial force set up in the beam when it is constrained to lie along the x-axis. The model for the deflection $u(x, t)$ which we discuss is

$$\ddot{u} + \alpha u^{(4)} - (\beta + k \mid u^{(1)} \mid^2)\, u^{(2)} = 0, \tag{3.1}$$

where $\alpha = EI/\rho$, $\beta = H/\rho$, $k = EA_c/2\rho l$, and $H = EA\Delta/l$, where E is the Young's modulus, I the cross-sectional second moment of area, ρ the density and A_c the cross-sectional area. We adopt the convention that if H is positive it represents a tensile force. Clearly $\alpha > 0$ and $k > 0$; these conditions are essential for the proofs which follow.

The initial conditions are

$$u(x, 0) = u_0(x), \tag{3.2a}$$

$$\dot{u}(x, 0) = u_1(x). \tag{3.2b}$$

In Section 4 we consider the boundary conditions corresponding to hinged ends

$$u(0, t) = u(l, t) = u^{(2)}(0, t) = u^{(2)}(l, t) = 0, \tag{3.3}$$

while in Section 5 we consider the boundary conditions corresponding to clamped ends,

$$u(0, t) = u(l, t) = u^{(1)}(0, t) = u^{(1)}(l, t) = 0. \tag{3.4}$$

All the solutions whose existence we prove satisfy the energy equation

$$\mid \dot{u} \mid^2 + \alpha \mid u^{(2)} \mid^2 + \beta \mid u^{(1)} \mid^2 + (k/2) \mid u^{(1)} \mid^4 = h, \tag{3.5}$$

where

$$h = \mid u_1 \mid^2 + \alpha \mid u_0^{(2)} \mid^2 + \beta \mid u_0^{(1)} \mid^2 + (k/2) \mid u_0^{(1)} \mid^4. \tag{3.6}$$

Consider the functional

$$G(u) = \frac{\mid u^{(2)} \mid^2}{\mid u^{(1)} \mid^2}, \tag{3.7}$$

where u is subject either to (3.3) or (3.4) and is supposed to be twice continuously differentiable in x. Then it is well known (Courant–Hilbert [9]) that $G(u)$ attains its minimum value, which is π^2/l^2 in the case (3.3) and $4\pi^2/l^2$ in the case (3.4).

Denote by $H_{\text{crit.}}$ the classical Euler buckling load of the beam.

$$H_{\text{crit.}} = -EI\pi^2/l^2 \qquad \text{for hinged ends,} \tag{3.8a}$$

while

$$H_{\text{crit.}} = -4EI\pi^2/l^2 \qquad \text{for clamped ends.} \tag{3.8b}$$

Then it is clear that if $H \geqslant H_{\text{crit.}}$ then $h \geqslant 0$, while if $H < H_{\text{crit.}}$ then there are initial conditions which allow h to be positive, negative, or zero.

The case $h < 0$ corresponds to motion about a buckled state, for $| u^{(1)} |$ cannot be zero in this case.

4. Hinged Ends

In this section we establish the existence of weak solutions of the equation (3.1) subject to the initial conditions (3.2) and the boundary conditions (3.3). We prove that the weak solutions are unique, satisfy the energy equation (3.5) and depend continuously on the initial data. We then prove that when the initial data is smooth enough and satisfies certain compatibility conditions the solution is a classical one. Precise meanings to the terms "weak solution" and "classical solution" are given in the statements of the theorems. As general references we cite Lions [19, pp. 1–26] and Sather [24], where the method is applied to a nonlinear hyperbolic equation. The results of this section include those of Dickey.

(i) *Definitions and Preliminary Lemmas*

Define

$$S_0 = \{y \in H^6(\Omega) \mid y, y^{(2)}, y^{(4)} \in H_0^1(\Omega)\},$$
$$S_1 = \{y \in H^4(\Omega) \mid y, y^{(2)} \in H_0^1(\Omega)\},$$
$$S_2 = H_0^1(\Omega) \cap H^2(\Omega).$$

S_0 and S_1 are easily seen to be complete subspaces of the Hilbert spaces $H^6(\Omega)$ and $H^4(\Omega)$ respectively.

Lemma 1. *Let $f \in H^1(\Omega)$ and suppose $f(\xi) = 0$ for some $\xi \in \bar{\Omega}$.*

Then

$$| f | \leqslant l/\sqrt{2} \, | f^{(1)} | . \tag{4.1}$$

Proof. By the Sobolev embedding theorem (see Friedman [15, p. 30]), f can be regarded as belonging to $C^0(\bar{\Omega})$ and hence $f(\xi) = 0$ has a meaning. Suppose $f \in C^1(\bar{\Omega})$. Then

$$f(x) = \int_\xi^x f^{(1)}(s)\, ds \qquad \text{for } x \in \Omega.$$

So

$$(f(x))^2 = \left(\int_\xi^x f^{(1)}(s)\, ds\right)^2 \leqslant \left| \int_\xi^x 1^2\, ds \right| \left| \int_\xi^x (f^{(1)}(s))^2\, ds \right|.$$

Therefore

$$|f|^2 \leqslant \int_0^l |x - \xi|\, dx\, |f^{(1)}|^2 \leqslant \frac{l^2}{2} |f^{(1)}|^2.$$

For a general $f \in H^1(\Omega)$, (4.1) follows by an approximation argument. □

By a *basis* of a Banach space X, we mean a set of linearly independent elements of X whose finite linear combinations are dense in X.

LEMMA 2. *$s_n = \sin(n\pi x/l)$, $n = 1, 2,\ldots$, is a basis of the spaces S_0, S_1, S_2 and $L^2(\Omega)$.*

Proof. That $\{s_n\}$ is a basis of $L^2(\Omega)$ is well known. Suppose $s \in S_0$ and let $\epsilon > 0$ be given. As $s^{(6)} \in L^2(\Omega)$, there exist N, $a_1,\ldots, a_N$ such that

$$\left| s^{(6)} - \sum_{r=1}^{N} a_r s_r \right|^2 < \epsilon.$$

Let

$$\varphi(x) = -\sum_{r=1}^{N} a_r \left(\frac{l}{r\pi}\right)^6 s_r(x)$$

so that $|(s - \varphi)^{(6)}|^2 < \epsilon$. By the Sobolev embedding theorem, s belongs to $C^5(\bar{\Omega})$, and so by Rolle's theorem there exist $\xi_i \in \bar{\Omega}$ such that

$$(s - \varphi)^{(i)}(\xi_i) = 0, \qquad 0 \leqslant i \leqslant 5.$$

Lemma 1 now implies that

$$\sum_{r=0}^{6} |(s - \varphi)^{(r)}|^2 < C\epsilon.$$

Hence $\{s_n\}$ is a basis of S_0; similarly $\{s_n\}$ is a basis of S_1 and of S_2. □

LEMMA 3 (Gronwall). *Suppose* $f \in L^\infty(0, T)$ *and that* $K \geqslant 0$, C_0 *are constants. If*

$$f(t) \leqslant C_0 + K \int_0^t f(s)\, ds$$

for all $t \in [0, T]$ *then*

$$f(t) \leqslant C_0 e^{Kt}.$$

Proof. See Carroll [7, p. 124]. □

LEMMA 4. *Suppose X and Y are Hilbert spaces or separable Banach spaces with dual spaces X′ and Y′. Suppose Y is continuously and densely embedded in X. If*

$$u_\mu \to u \quad \textit{in} \quad L^\infty(0, T; X') \textit{ weak}*$$

and

$$\dot{u}_\mu \to \chi \quad \textit{in} \quad L^\infty(0, T; Y') \textit{ weak}*,$$

then

$$\chi = \dot{u} \quad \textit{in} \quad L^\infty(0, T; Y').$$

Proof. The assumptions on X imply that $L^\infty(0, T; X')$ is the dual space of $L^1(0, T; X)$ (see Bochner and Taylor [4] and Dieudonné [12]) and that

$$\int_0^T u\,(t)\,(g(t))\, dt \to \int_0^T u(t)\,(g(t))\, dt \qquad \text{for all } g \in L^1(0, T; X).$$

Thus for all $x \in X$, $\varphi \in \mathscr{D}(]0, T[)$

$$\int_0^T \varphi(t) u_\mu(t)\,(x)\, dt \to \int_0^T \varphi(t) u(t)\,(x)\, dt.$$

Hence

$$u_\mu(\varphi)\,(x) \to u(\varphi)\,(x)$$

(using (2.2)) and so

$$\dot{u}_\mu(\varphi)\,(x) \to \dot{u}(\varphi)\,(x).$$

Similarly, $\dot{u}_\mu(\varphi)\,(y) \to \chi(\varphi)\,(y)$ for all $y \in Y$ and $\varphi \in \mathscr{D}(]0, T[)$.

Thus $\dot{u}(\varphi) = \chi(\varphi)$ in Y' for all $\varphi \in \mathscr{D}(]0, T[)$ and the result follows. □

LEMMA 5. *Let X be a Banach space. If* $f \in L^2(0, T; X)$ *and* $\dot{f} \in L^2(0, T; X)$, *then f, possibly after redefinition on a set of measure zero, is continuous from* $[0, T] \to X$. *Indeed, for almost all* $s, t \in [0, T]$,

$$f(t) - f(s) = \int_s^{t_0} \dot{f}(\sigma)\, d\sigma.$$

Proof. See Wilcox [27, Theorem 2.2]. A similar lemma, due to Lions, is proved in Carroll [7, p. 176]. □

The next lemma establishes a monotonicity property for the nonlinear term in (3.1).

LEMMA 6. *If $u, v \in S_2$ then*

$$(|u^{(1)}|^2 u^{(2)} - |v^{(1)}|^2 v^{(2)}, u - v) \leqslant 0.$$

Proof.

$$\begin{aligned}
&(|u^{(1)}|^2 u^{(2)} - |v^{(1)}|^2 v^{(2)}, u - v)\\
&\quad = |u^{(1)}|^2 ((u^{(1)}, v^{(1)}) - |u^{(1)}|^2) + |v^{(1)}|^2 ((u^{(1)}, v^{(1)}) - |v^{(1)}|^2)\\
&\quad \leqslant |u^{(1)}|^2 (|u^{(1)}|\,|v^{(1)}| - |u^{(1)}|^2) + |v^{(1)}|^2 (|u^{(1)}|\,|v^{(1)}| - |v^{(1)}|^2)\\
&\quad = -(|u^{(1)}| - |v^{(1)}|)(|u^{(1)}|^3 - |v^{(1)}|^3) \leqslant 0. \quad \square
\end{aligned}$$

(ii) *Weak Solutions*

In this subsection we establish the existence of a weak solution to the initial-boundary value problem (3.1)–(3.3). The solution need not satisfy the boundary conditions $u^{(2)}(0, t) = u^{(2)}(l, t) = 0$ in any classical sense, although we shall show later (Theorem 4) that it does do so if u_0 and u_1 are smooth enough and if

$$u_0^{(2)}(0) = u_0^{(2)}(l) = u_1(0) = u_1(l) = 0.$$

THEOREM 1. *If $u_0 \in S_2$, $u_1 \in L^2(\Omega)$, then there exists $u \equiv u(x, t)$ with*

$$u \in L^\infty(0, T; S_2),$$
$$\dot{u} \in L^\infty(0, T; L^2(\Omega)),$$

such that u satisfies the initial conditions (3.2) *and the equation* (3.1) *in the sense that*

$$(\ddot{u}, \varphi) + \alpha(u^{(2)}, \varphi^{(2)}) - (\beta + k\,|u^{(1)}|^2)(u^{(2)}, \varphi) = 0 \qquad \text{for all } \varphi \in S_2. \quad (4.2)$$

Proof.

Approximating solutions. Let $\{w_j\}$ be a basis of S_2. If

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, w_i$$

is to be a solution of

$$(\ddot{u}_m(t), w_j) + \alpha(u_m^{(2)}(t), w_j^{(2)}) - (\beta + k \mid u_m^{(1)}(t)|^2)(u_m^{(2)}(t), w_j) = 0 \qquad 1 \leqslant j \leqslant m, \tag{4.3}$$

certain nonlinear ordinary differential equations for the g_{im} must be satisfied. These can be written in the form

$$\ddot{g}_{im} + \sum_{j=1}^{m} (w^{-1})_{ij} f_{jm}(g) = 0 \qquad 1 \leqslant i \leqslant m, \tag{4.4}$$

where $w = (w_{ij})$, $w_{ij} = (w_i, w_j)$, $g = (g_{ij})$ and f_{jm} is continuous. w^{-1} exists by the assumed linear independence of $\{w_j\}$.

The method of successive approximations (see Coddington and Levinson [8, Chapter 1]) ensures the existence of a solution u_m to (4.4) and thus to (4.3), valid in $[0, t_m]$, subject to the initial conditions

$$\begin{aligned} u_m(0) &= u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \to u_0 \qquad \text{in} \qquad S_2 \\ \dot{u}_m(0) &= u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \to u_1 \qquad \text{in} \qquad L^2(\Omega), \end{aligned} \tag{4.5}$$

where we have used the assumptions on u_0 and u_1.

The estimates that follow show among other things that $t_m = T$.

Estimates. Multiply (4.3) by $\dot{g}_{jm}(t)$ and sum for $j = 1,..., m$. This gives

$$\frac{d}{dt}\left(\frac{1}{2} \mid \dot{u}_m \mid^2 + \frac{\alpha}{2} \mid u_m^{(2)} \mid^2 + \frac{\beta}{2} \mid u_m^{(1)} \mid^2 + \frac{k}{4} \mid u_m^{(1)} \mid^4\right) = 0.$$

Integrating from 0 to t yields the energy equation

$$\begin{aligned} &\mid \dot{u}_m \mid^2 + \alpha \mid u_m^{(2)} \mid + \beta \mid u_m^{(1)} \mid^2 + \frac{k}{2} \mid u_m^{(1)} \mid^4 \\ &\quad = \mid u_{1m} \mid^2 + \alpha \mid u_{0m}^{(2)} \mid^2 + \beta \mid u_{0m}^{(1)} \mid^2 + \frac{k}{2} \mid u_{0m}^{(1)} \mid^4. \end{aligned} \tag{4.6}$$

The right hand side of (4.6) is bounded independent of m and t [from (4.5)] and as $\alpha, k > 0$ it is clear that

$$\mid u_m^{(1)} \mid, \mid u_m^{(2)} \mid, \mid \dot{u}_m \mid < C \qquad \text{(independent of } m \text{ and } t). \tag{4.7}$$

Convergence. The estimates just derived, together with Lemma 1, show that

$$\{u_m\} \quad \text{is bounded in} \quad L^\infty(0, T; S_2),$$

$$\{\dot{u}_m\} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)),$$

and

$$\{| u_m^{(1)} |^2 u_m^{(2)}\} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)).$$

In particular, $\{u_m\}$ is bounded in $H^1(Q)$. Furthermore, the injection $H^1(Q) \to L^2(Q)$ is compact by the Rellich–Kondrachoff theorem (Lions and Magenes [20, p. 110]). Thus, using the classical diagonal procedure, we may extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ with the properties

$$\begin{aligned} u_\mu &\to u \quad &&\text{in} \quad L^\infty(0, T; S_2) \text{ weak*},\\ \dot{u}_\mu &\to v \quad &&\text{in} \quad L^\infty(0, T; L^2(\Omega)) \text{ weak*},\\ u_\mu &\to w \quad &&\text{in} \quad L^2(Q) \text{ strongly and a.e.,} \end{aligned} \tag{4.8}$$

and

$$| u_\mu^{(1)} |^2 u_\mu^{(2)} \to \chi \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).$$

Lemma 4 implies that $\dot{u} = v$. As $u_\mu \to u$ in $L^2(Q)$ weak* it follows that $u = w$. The next step is to show that

$$\chi = | u^{(1)} |^2 u^{(2)}. \tag{4.9}$$

To this end let $v \in L^2(0, T; S_2)$.

From Lemma 6 it follows that

$$\int_0^T (| u_\mu^{(1)} |^2 u_\mu^{(2)} - | v^{(1)} |^2 v^{(2)}, u_\mu - v)\, dt \leqslant 0.$$

But

$$\int_0^T (| u_\mu^{(1)} |^2 u_\mu^{(2)}, u_\mu)\, dt = \int_0^T (| u_\mu^{(1)} |^2 u_\mu^{(2)}, u)\, dt + \int_0^T (| u_\mu^{(1)} |^2 u_\mu^{(2)}, u_\mu - u)\, dt.$$

As $\mu \to \infty$, the first integral on the right hand side converges to $\int_0^T (\chi, u)\, dt$, while the second integral tends to zero since $u_\mu \to u$ in $L^2(Q)$ strongly.

Hence as

$$\int_0^T (| v^{(1)} |^2 v^{(2)}, u_\mu - v)\, dt \to \int_0^T (| v^{(1)} |^2 v^{(2)}, u - v)\, dt,$$

it follows that

$$\int_0^T (\chi - | v^{(1)} |^2 v^{(2)}, u - v)\, dt \leqslant 0.$$

Set $v = u - \lambda w$ with $\lambda > 0$ and $w \in L^2(0, T; S_2)$, and let $\lambda \to 0^+$. Thus

$$\int_0^T (\chi - |u^{(1)}|^2 u^{(2)}, w)\, dt \leqslant 0.$$

Change w to $-w$. Then

$$\int_0^T (\chi - |u^{(1)}|^2 u^{(2)}, w)\, dt = 0 \qquad \text{for all } w \in L^2(0, T; S_2).$$

Hence

$$\chi = |u^{(1)}|^2 u^{(2)}.$$

Now let j be fixed and $\mu > j$. From (4.8) and (4.9) it follows that

$$(u_\mu^{(2)}, w_j^{(2)}) \to (u^{(2)}, w_j^{(2)}) \qquad \text{in} \qquad L^\infty(0, T) \text{ weak}^*,$$
$$(u_\mu^{(2)}, w_j) \to (u^{(2)}, w_j) \qquad \text{in} \qquad L^\infty(0, T) \text{ weak}^*,$$
$$(|u_\mu^{(1)}|^2 u_\mu^{(2)}, w_j) \to (|u^{(1)}|^2 u^{(2)}, w_j) \qquad \text{in} \qquad L^\infty(0, T) \text{ weak}^*.$$

Also, since

$$(\dot{u}_\mu, w_j) \to (\dot{u}, w_j) \qquad \text{in} \qquad L^\infty(0, T) \text{ weak}^*,$$
$$(\ddot{u}_\mu, w_j) \to (\ddot{u}, w_j) \qquad \text{in} \qquad \mathscr{D}'(]0, T[).$$

Hence

$$(\ddot{u}, w_j) + \alpha(u^{(2)}, w_j^{(2)}) - (\beta + k\,|u^{(1)}|^2)(u^{(2)}, w_j) = 0 \tag{4.10}$$

and (4.2) follows from the denseness in S_2 of the basis $\{w_j\}$.

It remains to show that the initial conditions (3.2) are satisfied by u. As

$$u_\mu \to u \qquad \text{in} \qquad L^\infty(0, T; L^2(\Omega)) \text{ weak}^*,$$

and

$$\dot{u}_\mu \to \dot{u} \qquad \text{in} \qquad L^\infty(0, T; L^2(\Omega)) \text{ weak}^*,$$

it follows from Lemma 5 that

$$(u_{\mu 0}, \varphi) \to (u(0), \varphi) \qquad \text{for all } \varphi \in L^2(\Omega),$$

and hence from (4.5)

$$u(0) = u_0.$$

From (4.10),

$$(\ddot{u}_\mu, w_j) \to (\ddot{u}, w_j) \qquad \text{in} \qquad L^\infty(0, T) \text{ weak}^*.$$

Thus from Lemma 5 with $X = \mathbb{R}$,

$$(\dot{u}_\mu(0), w_j) \to (\dot{u}(0), w_j).$$

But

$$(\dot{u}_\mu(0), w_j) \to (u_1, w_j)$$

and so

$$\dot{u}(0) = u_1. \qquad \square$$

Remark. We sketch another proof of the convergence of the nonlinear term, following that of Dickey, and using the special form of the term. For $\varphi \in L^1(0, T; L^2(\Omega))$ we have that

$$\begin{aligned}\int_0^T (\chi - | u^{(1)} |^2 u^{(2)}, \varphi)\, dt &= \int_0^T (\chi - | u_\mu^{(1)} |^2 u_\mu^{(2)}, \varphi)\, dt + \int_0^T | u^{(1)} |^2 (u_\mu^{(2)} - u^{(2)}, \varphi)\, dt \\ &\quad + \int_0^T (| u_\mu^{(1)} |^2 - | u^{(1)} |^2)(u_\mu^{(2)}, \varphi)\, dt.\end{aligned}$$

But

$$\begin{aligned}\left| \int_0^T (| u_\mu^{(1)} |^2 - | u^{(1)} |^2)(u^{(2)}, \varphi)\, dt \right| &\leqslant C \left| \int_0^T (u_\mu - u, u_\mu^{(2)} + u^{(2)})\, dt \right| \\ &\leqslant C \int_0^T | u_\mu - u |\, dt \to 0.\end{aligned}$$

The other integrals also tend to zero and the arbitrariness of φ implies that

$$\chi = | u^{(1)} |^2 u^{(2)}.$$

(iii) *Dependence on Initial Conditions*

Next we show that the solution u in Theorem 1 satisfies the energy equation (3.5), and that u depends continuously on the initial data u_0 and u_1. In particular we prove that u is unique. The following lemma is a special case of Lemma 8.3, p. 298 of Lions and Magenes [20], originally due to Strauss [25]. We omit the proof, which relies on an intricate regularization procedure.

LEMMA 7. *Let V be continuously and densely embedded in $L^2(\Omega)$ and let V' be the dual of V so that $V \subset L^2(\Omega) \subset V'$. If $\psi \in V$ define $\Delta^2\psi \in V'$ by*

$\Delta^2\psi(\varphi) = \alpha(\psi^{(2)}, \varphi^{(2)})$ for $\varphi \in V$, so that $\Delta^2 \in \mathscr{L}(V, V')$. Suppose $w \in L^\infty(0, T; V)$, $\dot{w} \in L^\infty(0, T; L^2(\Omega))$ and that w satisfies the equation

$$\ddot{w} + \Delta^2 w = f,$$

and the initial conditions $w(0) = w_0$, $\dot{w}(0) = w_1$. Suppose $w_0 \in V$, $w_1 \in L^2(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$. Then for all $t \in [0, T]$,

$$|\dot{w}(t)|^2 + \alpha |w^{(2)}(t)|^2 = |w_1|^2 + \alpha |w_0^{(2)}|^2 + 2\int_0^t (f, \dot{w})\, d\sigma. \tag{4.11}$$

THEOREM 2. *Suppose u, v are two solutions of (4.2) with*

$$u, v \in L^\infty(0, T; S_2),$$
$$\dot{u}, \dot{v} \in L^\infty(0, T; L^2(\Omega)),$$

and suppose that u, v satisfy the initial conditions

$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad v(0) = v_0, \qquad \dot{v}(0) = v_1,$$

with u_0, $v_0 \in S_2$ and u_1, $v_1 \in L^2(\Omega)$. Let $w = u - v$. Then

$$|\dot{w}(t)|^2 + \alpha |w^{(2)}(t)|^2 \leqslant [|u_1 - v_1|^2 + \alpha |u_0^{(2)} - v_0^{(2)}|^2] \exp(Kt), \tag{4.12}$$

where K is a continuous function of $|u_0^{(2)}|$, $|u_1|$, $|v_0^{(2)}|$ and $|v_1|$.

Proof. We apply Lemma 7 with $V = S_2$, $w_0 = u_0 - v_0$, $w_1 = u_1 - v_1$ and

$$f(t) = (\beta + k |u^{(1)}(t)|^2)\, u^{(2)}(t) - (\beta + k |v^{(1)}(t)|^2)\, v^{(2)}(t).$$

It is clear that $f \in L^2(0, T; L^2(\Omega))$ and we conclude that

$$|\dot{w}(t)|^2 + \alpha |w^{(2)}(t)|^2 = |u_1 - v_1|^2 + \alpha |u_0^{(2)} - v_0^{(2)}|^2 + 2\int_0^t (f, \dot{w})\, d\sigma. \tag{4.13}$$

But

$$\begin{aligned} f(\sigma) &= (\beta + k |u^{(1)}(\sigma)|^2)\, w^{(2)}(\sigma) + k(|u^{(1)}(\sigma)|^2 - |v^{(1)}(\sigma)|^2)\, v^{(2)}(\sigma) \\ &= (\beta + k |u^{(1)}(\sigma)|^2)\, w^{(2)}(\sigma) - k(u(\sigma) + v(\sigma), w^{(2)}(\sigma))\, v^{(2)}(\sigma) \end{aligned}$$

and hence

$$\begin{aligned} \left| 2\int_0^t (f, \dot{w})\, d\sigma \right| &\leqslant C \int_0^t |w^{(2)}(\sigma)|\, |\dot{w}(\sigma)|\, d\sigma \\ &\leqslant \frac{C}{2\alpha^{1/2}} \int_0^t (|\dot{w}(\sigma)|^2 + \alpha |w^{(2)}(\sigma)|^2)\, d\sigma. \end{aligned}$$

(4.12) now follows from (4.13) and Lemma 3.

That $K = C/2\alpha^{1/2}$ is a continuous function of $|u_1|$, $|v_1|$, $|u_0^{(2)}|$ and $|v_0^{(2)}|$ is a consequence of Lemma 1 and Theorem 3 to follow. □

Remarks. (i) It is an immediate consequence of the theorem that if $u_0 = v_0$ and $u_1 = v_1$, then $u = v$; that is, the solution u in Theorem 1 is unique. Choosing subsequences in Theorem 1 was therefore unnecessary, and our method is therefore constructive. The uniqueness may also be proved directly (avoiding Lemma 7) by the method given by Lions [19, p. 15].

(ii) Inequality (4.12) may be interpreted as an estimate for the error in the solution when the basis $\{\sin(j\pi x/l)\}$ (in any order) is used, as in this case the approximate solutions satisfy (3.1). In (4.12) set $v = u_m$, $v_0 = u_{0m}$, $v_1 = u_{1m}$. The upper bound for the error is then seen to depend on how well the initial data is approximated, and it increases exponentially with time as might be expected for an undamped system. K may be evaluated explicitly in terms of α, β, k, u_{0m} and u_{1m}.

THEOREM 3. *The unique solution in Theorem* 1 *satisfies the energy equation* (3.5).

Proof. Set $v \equiv 0$ in (4.13). It is then enough to prove that

$$2\int_0^t (\beta + k\,|u^{(1)}(\sigma)|^2)\,(u^{(2)}(\sigma), \dot{u}(\sigma))\,d\sigma = \beta(|u_0^{(1)}|^2 - |u^{(1)}(t)|^2) + \frac{k}{2}(|u_0^{(1)}|^4 - |u^{(1)}(t)|^4). \tag{4.14}$$

From Lemma 5 it follows that $u(t)$ is a strongly absolutely continuous function of t with values in $L^2(\Omega)$. (See Hille and Phillips [17, p. 83].)

But

$$\big|\,|u^{(1)}(s)|^2 - |u^{(1)}(t)|^2\,\big| = |(u(s) - u(t), u^{(2)}(s) - u^{(2)}(t))| \leqslant C\,|u(s) - u(t)|.$$

Hence $|u^{(1)}(t)|^2$ is a real valued absolutely continuous function of t with derivative $-2(u^{(2)}(t), \dot{u}(t))$ a.e. Similarly, $|u^{(1)}(t)|^4$ is absolutely continuous with derivative $-4\,|u^{(1)}(t)|^2\,(u^{(2)}(t), \dot{u}(t))$ a.e. Thus (4.14) holds. □

(iv) *Regularity*

THEOREM 4. *Suppose $u_0 \in S_1$ and $u_1 \in S_2$. Then there exists a unique function u such that*

$$\begin{gathered} u \in L^\infty(0, T; S_1), \qquad \dot{u} \in L^\infty(0, T; S_2), \qquad \ddot{u} \in L^\infty(0, T; L^2(\Omega)), \\ \ddot{u} + \alpha u^{(4)} - (\beta + k\,|u^{(1)}|^2)\,u^{(2)} = 0 \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)), \\ u(0) = u_0, \quad \text{and} \quad \dot{u}(0) = u_1. \end{gathered} \tag{4.15}$$

Proof. The proof closely follows that of Theorem 1.

Approximating solutions. We use the basis $\{s_j\} = \{\sin(j\pi x/l)\}$ of S_1. (See Lemma 2). The approximating solutions u_m are of the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, s_i \tag{4.16}$$

and satisfy the equation

$$\ddot{u}_m + \alpha u_m^{(4)} - (\beta + k \mid u_m^{(1)} \mid^2)\, u_m^{(2)} = 0 \tag{4.17}$$

in $[0, t_m]$ subject to the initial conditions

$$\begin{aligned} u_m(0) &= u_{0m} \to u_0 \quad &\text{in} \quad & S_1 \\ \dot{u}_m(0) &= u_{1m} \to u_1 \quad &\text{in} \quad & S_2 . \end{aligned} \tag{4.18}$$

Estimates. The basic energy estimate (4.7) holds as before, and shows that $t_m = T$. It follows from (4.17) and (4.18) that

$$\mid \ddot{u}_m(0) \mid < C. \tag{4.19}$$

Now differentiate (4.17) with respect to t and take the inner product with $\ddot{u}_m$, to obtain

$$\begin{aligned} \tfrac{1}{2}\,(d/dt)\,(\mid \ddot{u}_m \mid^2 + \alpha \mid \dot{u}^{(2)} \mid^2) &= (\beta \dot{u}_m^{(2)} + k \mid u_m^{(1)} \mid^2 \dot{u}_m^{(2)} + 2k(u_m^{(1)}, \dot{u}_m^{(1)})\, u_m^{(2)}, \ddot{u}_m) \\ &\leqslant (\mid \beta \mid + k \mid u_m^{(1)} \mid^2) \mid \dot{u}_m^{(2)} \mid \mid \ddot{u}_m \mid \\ &\quad + 2k \mid u_m^{(1)} \mid \mid \dot{u}_m^{(1)} \mid \mid u_m^{(2)} \mid \mid \ddot{u}_m \mid . \end{aligned}$$

By Lemma 1 and (4.7)

$$\begin{aligned} (d/dt)\,(\mid \ddot{u}_m \mid^2 + \alpha \mid \dot{u}_m^{(2)} \mid^2) &\leqslant C \mid \dot{u}_m^{(2)} \mid \mid \ddot{u}_m \mid \\ &\leqslant \frac{C}{2\alpha^{1/2}} (\mid \ddot{u}_m \mid^2 + \alpha \mid \dot{u}_m^{(2)} \mid^2). \end{aligned}$$

It follows from (4.19) and Lemma 3 that

$$\mid \ddot{u}_m \mid , \mid \dot{u}_m^{(2)} \mid < C \qquad \text{(independent of } m \text{ and } t). \tag{4.20}$$

As $\alpha > 0$, (4.17) yields the bound

$$\mid u_m^{(4)} \mid < C. \tag{4.21}$$

Convergence. Using the estimates just derived, Lemma 1 and the methods of Theorem 1, it is easy to show the existence of a subsequence $\{u_\mu\}$ of $\{u_m\}$ such that

$$
\begin{aligned}
u_\mu &\to u && \text{in} \quad L^\infty(0, T; S_1) \text{ weak*},\\
\dot{u}_\mu &\to \dot{u} && \text{in} \quad L^\infty(0, T; S_2) \text{ weak*},\\
\ddot{u}_\mu &\to \ddot{u} && \text{in} \quad L^\infty(0, T; L^2(\Omega)) \text{ weak*},\\
u_\mu &\to u && \text{in} \quad H^1(Q) \text{ strongly and a.e.,}
\end{aligned}
$$

and

$$| u_\mu^{(1)} |^2 u_\mu^{(2)} \to | u^{(1)} |^2 u^{(2)} \qquad \text{in} \quad L^\infty(0, T; S_2) \text{ weak*}.$$

These convergence properties establish the theorem. The proof parallels exactly that of Theorem 1. u is unique by Theorem 2. □

Remark. By the Sobolev embedding theorem, u_0 and $u(t)$ are equivalent to functions in $C^3(\bar{\Omega})$ and, therefore, (Friedman [15, p. 39]) satisfy the hinged-end boundary conditions (3.3). Similarly $u(0, t) = u(l, t) = 0$ for all $t \in [0, T]$. The embedding theorem also shows that $u \in C^0(Q)$.

The next theorem establishes that under certain conditions u is a *classical solution*; that is $u \in C^4(\bar{\Omega}) \times C^2([0, T])$ and satisfies (3.1)–(3.3). By putting $x = 0, l$ and $t = 0$ in (3.1) it is clear that necessary conditions for the existence of a classical solution are the "compatibility conditions" $u_0^{(4)}(0) = u_0^{(4)}(l) = 0$. Roughly speaking, the theorem says that these conditions are sufficient.

THEOREM 5. *Let $u_0 \in S_0$ and $u_1 \in S_1$. Then*

$$
\begin{gathered}
u \in L^\infty(0, T; S_0), \qquad \dot{u} \in L^\infty(0, T; S_1), \qquad \ddot{u} \in L^\infty(0, T; S_2),\\
\dddot{u} \in L^\infty(0, T; L^2(\Omega)) \qquad \text{and} \qquad u \in C^1(\bar{Q}) \cap [C^5(\bar{\Omega}) \times C^2([0, T])].
\end{gathered}
\tag{4.22}
$$

Proof. As in Theorem 4 we use the basis $\{s_j\}$ of S_0. The approximating solutions u_m are of the form (4.16) and satisfy (4.17) in $[0, t_m]$ subject to the initial conditions

$$
\begin{aligned}
u_m(0) &= u_{0m} \to u_0 && \text{in} \quad S_0,\\
\dot{u}_m(0) &= u_{1m} \to u_1 && \text{in} \quad S_1.
\end{aligned}
$$

The bounds (4.7), (4.20) and (4.21) hold and show that $t_m = T$. Taking the inner product of (4.17) with $\dot{u}_m^{(8)}(t)$ leads to

$$
\begin{aligned}
\tfrac{1}{2}(d/dt)(| \dot{u}_m^{(4)} |^2 + \alpha | u_m^{(6)} |^2) &= (\beta + k | u_m^{(1)} |^2)(u_m^{(6)}, \dot{u}_m^{(4)})\\
&\leqslant C(| \dot{u}_m^{(4)} |^2 + \alpha | u_m^{(6)} |^2).
\end{aligned}
$$

Thus $|\dot{u}_m^{(4)}|$, $|u_m^{(6)}| < C$ (independent of m and t), and from (4.17) it follows that $|\ddot{u}_m^{(2)}|$, $|\dddot{u}_m| < C$. Hence, using Lemma 1, we may extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ such that

$$\begin{aligned} u_\mu &\to u && \text{in} && L^\infty(0, T; S_0) \text{ weak}^*, \\ \dot{u}_\mu &\to \dot{u} && \text{in} && L^\infty(0, T; S_1) \text{ weak}^*, \\ \ddot{u}_\mu &\to \ddot{u} && \text{in} && L^\infty(0, T; S_2) \text{ weak}^*, \\ \dddot{u}_\mu &\to \dddot{u} && \text{in} && L^\infty(0, T; L^2(\Omega)) \text{ weak}^*, \\ u_\mu &\to u && \text{in} && H^2(Q) \text{ strongly and a.e.,} \end{aligned}$$

and

$$|u_\mu^{(1)}|^2 u_\mu^{(2)} \to |u^{(1)}|^2 u^{(2)} \quad \text{in} \quad L^\infty(0, T; S_1) \text{ weak}^*.$$

The proof is completed as in Theorem 1; (4.22) follows from the embedding theorems, since, for example, $u \in H^3(Q)$. □

Remark. As in the remark after Theorem 4, we may show that

$$u(0, t) = u(l, t) = u^{(2)}(0, t) = u^{(2)}(l, t) = u^{(4)}(0, t) = u^{(4)}(l, t) = 0,$$

that

$$\dot{u}(0, t) = \dot{u}(l, t) = \dot{u}^{(2)}(0, t) = \dot{u}^{(2)}(l, t) = 0$$

and that

$$\ddot{u}(0, t) = \ddot{u}(l, t) = 0 \qquad \text{for all } t \in [0, T].$$

5. Clamped Ends

In this section we study the initial-boundary value problem for the equation (3.1) subject to the initial conditions (3.2) and the boundary conditions for clamped ends (3.4).

(i) *Weak Solutions*

First of all we prove the existence of a weak solution and its continuous dependence on the initial data. We also show that the weak solution is unique and satisfies the energy equation (3.5).

Theorem 6. *If $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, then there exists $u \equiv u(x, t)$ with*

$$u \in L^\infty(0, T; H_0^2(\Omega)),$$
$$\dot{u} \in L^\infty(0, T; L^2(\Omega)),$$

such that u satisfies the initial conditions (3.2) *and the equation* (3.1) *in the sense that*

$$(\ddot{u}, \varphi) + \alpha(u^{(2)}, \varphi^{(2)}) - (\beta + k \mid u^{(1)} \mid^2)(u^{(2)}, \varphi) = 0 \qquad \textit{for all } \varphi \in H_0^2(\Omega). \tag{5.1}$$

Proof. The proof is practically identical to that of Theorem 1. We start with a basis $\{w_j\}$ of $H_0^2(\Omega)$ and establish the existence of approximating solutions

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\, w_i \tag{5.2}$$

to the equations

$$(\ddot{u}_m(t), w_j) + \alpha(u_m^{(2)}, w_j^{(2)}) - (\beta + k \mid u_m^{(1)}(t)\mid^2)(u_m^{(2)}, w_j) = 0 \qquad 1 \leqslant j \leqslant m \tag{5.3}$$

subject to the initial conditions

$$\begin{aligned} u_m(0) = u_{0m} &= \sum_{i=1}^{m} \alpha_{1m} w_i \to u_0 \qquad \text{in} \qquad H_0^2(\Omega) \\ \dot{u}_m(0) = u_{1m} &= \sum_{i=1}^{m} \beta_{im} w_i \to u_1 \qquad \text{in} \qquad L^2(\Omega). \end{aligned} \tag{5.4}$$

The energy bound

$$\mid u_m^{(1)} \mid, \mid u_m^{(2)} \mid, \mid \dot{u}_m \mid < C \tag{5.5}$$

still holds and the rest of the proof goes through in a straightforward way. □

Remark. u satisfies the boundary conditions (3.4) by the embedding theorems.

THEOREM 7. *Suppose u, v are two solutions of* (5.1) *with*

$$u, v \in L^\infty(0, T; H_0^2(\Omega)),$$
$$\dot{u}, \dot{v} \in L^\infty(0, T; L^2(\Omega))$$

and suppose that u, v satisfy the initial conditions

$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad v(0) = v_0, \qquad \dot{v}(0) = v_1$$

with

$$u_0, v_0 \in H_0^2(\Omega) \qquad \textit{and} \qquad u_1, v_1 \in L^2(\Omega).$$

Set $w = u - v$. Then

$$|\dot{w}(t)|^2 + \alpha\,|w^{(2)}(t)|^2 \leqslant [|u_1 - v_1|^2 + \alpha\,|u_0^{(2)} - v_0^{(2)}|^2]\exp(K_1 t) \tag{5.6}$$

where K_1 is a continuous function of $|u_0^{(2)}|$, $|u_1|$, $|v_0^{(2)}|$ and $|v_1|$.

Proof. The proof runs parallel to that of Theorem 2. In applying Lemma 7 we set $V = H_0^2(\Omega)$. □

Remark. Setting $u_0 = v_0$ and $u_1 = v_1$ in (5.6) demonstrates the uniqueness of the weak solution in Theorem 6. The Galerkin method is therefore constructive. A direct proof of uniqueness can again be given following [19, p. 15].

THEOREM 8. *The unique solution in Theorem 6 satisfies the energy equation.*

Proof. Identical to that of Theorem 3. □

(ii) *Smoother Solutions*

This subsection contains a lemma and a preliminary regularity result. Let X be the Hilbert space $H_0^2(\Omega) \cap H^4(\Omega)$.

LEMMA 8. *There are constants C_i such that for all $f \in X$,*

$$|f^{(i)}| \leqslant C_i\,|f^{(i+1)}| \qquad i = 0, 1, 2, 3.$$

Proof. By the embedding theorems and Rolle's theorem, there exists ξ_1, $0 < \xi_1 < l$ with $f^{(1)}(\xi_1) = 0$. Therefore there exist ξ_2, ξ_3, $0 < \xi_2 < \xi_1 < \xi_3 < l$ with $f^{(2)}(\xi_2) = f^{(2)}(\xi_3) = 0$, and there exists ξ_4, $0 < \xi_2 < \xi_4 < \xi_3 < l$ with $f^{(3)}(\xi_4) = 0$. The result now follows from Lemma 1.

THEOREM 9. *If $u_0 \in X$, $u_1 \in H_0^2(\Omega)$ then there exists a unique function $u \equiv u(x, t)$ with*

$$u \in L^\infty(0, T; X),$$
$$\dot{u} \in L^\infty(0, T; H_0^2(\Omega)),$$
$$\ddot{u} \in L^\infty(0, T; L^2(\Omega)),$$

such that u satisfies the initial conditions (3.2) and the equation

$$\ddot{u} + \alpha u^{(4)} - (\beta + k\,|u^{(1)}|^2)\,u^{(2)} = 0 \qquad in \qquad L^\infty(0, T; L^2(\Omega)). \tag{5.7}$$

Proof. Let $\{w_j\}$ be a basis of X. The approximating solutions u_m are of the form (5.2) and satisfy (5.3), and the initial conditions

$$u_m(0) \equiv u_{m0} \to u_0 \qquad \text{in} \qquad X,$$

and

$$\dot{u}_m(0) \equiv u_{m1} \to u_1 \qquad \text{in} \qquad H_0^2(\Omega).$$

$\{u_m\}$ satisfies the bounds (5.5).

Multiply (5.3) by $\ddot{g}_{jm}(0)$ and sum for $j = 1, \ldots, m$. Thus

$$|\ddot{u}_m(0)|^2 = |(\alpha u_{m0}^{(4)} - (\beta + k \mid u_{m0}^{(1)} \mid^2)\, u_{m0}^{(2)}, \ddot{u}_m(0))| \leqslant C \mid \ddot{u}_m(0)| .$$

Thus

$$|\ddot{u}_m(0)| < C. \tag{5.8}$$

Now differentiate (5.3) with respect to t to obtain

$$\begin{aligned}(\dddot{u}_m, w_j) &+ \alpha(\dot{u}_m^{(2)}, w_j^{(2)}) \\ &= \beta(\dot{u}_m^{(2)}, w_j) + k[2(u_m^{(1)}, \dot{u}_m^{(1)})\,(u_m^{(2)}, w_j) + \mid u_m^{(1)} \mid^2 (\dot{u}_m^{(2)}, w_j)].\end{aligned} \tag{5.9}$$

Multiply (5.9) by $\ddot{g}_{jm}(t)$ and sum for $j = 1, \ldots, m$. It follows that

$$\begin{aligned}&\tfrac{1}{2}\,(d/dt)\,(\mid \ddot{u}_m \mid^2 + \alpha \mid \dot{u}_m^{(2)} \mid^2) \\ &\leqslant \mid \beta \mid \mid \dot{u}_m^{(2)} \mid \mid \ddot{u}_m \mid + 2k \mid u_m^{(1)} \mid \mid \dot{u}_m^{(1)} \mid \mid u_m^{(2)} \mid \mid \ddot{u}_m \mid + k \mid u_m^{(1)} \mid^2 \mid \dot{u}_m^{(2)} \mid \mid \ddot{u}_m \mid \\ &\leqslant C \mid \dot{u}_m^{(2)} \mid \mid \ddot{u}_m \mid \leqslant \frac{C}{2\alpha^{1/2}}\,(\mid \ddot{u}_m \mid^2 + \alpha \mid \dot{u}_m^{(2)} \mid^2).\end{aligned}$$

From (5.8) and Lemma 3 we deduce the bounds

$$\mid \ddot{u}_m \mid, \mid \dot{u}_m^{(2)} \mid < C. \tag{5.10}$$

In the usual way follows the existence of u with

$$u \in L^\infty(0, T; H_0^2(\Omega)), \qquad \dot{u} \in L^\infty(0, T; H_0^2(\Omega)), \qquad \ddot{u} \in L^\infty(0, T; L^2(\Omega)),$$

and such that

$$(\ddot{u}, v) + \alpha(u^{(2)}, v^{(2)}) - (\beta + k \mid u^{(1)} \mid^2)\,(u^{(2)}, v) = 0 \qquad \text{for all } v \in X. \tag{5.11}$$

It remains to show that $u \in L^\infty(0, T; X)$.

But from (5.11), for almost all t and for all $v \in X$, $\alpha(u^{(2)}, v^{(2)}) \in L^\infty(0, T)$.

Hence, for almost all t, u satisfies (3.1) and so $u^{(4)} \in L^\infty(0, T; L^2(\Omega))$. Lemma 8 now shows that $u \in L^\infty(0, T; X)$. □

(iii) *Regularity and an Associated Linear Equation*

If $u \in C^5(\bar{\Omega}) \times C^2([0, T])$, and if u satisfies (3.1), (3.2) and (3.4), then we call u a *classical solution* of the initial-boundary value problem. Setting $t = 0, x = 0, l$ in (3.1), we see that necessary conditions for a classical solution are that

$$\alpha u_0^{(4)} - (\beta + k \mid u_0^{(1)} \mid^2) u_0^{(2)} = \alpha u_0^{(5)} - (\beta + k \mid u_0^{(1)} \mid^2) u_0^{(3)} = 0 \quad \text{at} \quad x = 0, l. \tag{5.12}$$

To obtain a result in the other direction is less straightforward than in the hinged-end case. This is due to the nonlinearity of the compatibility conditions (5.12). In the hinged-end case (see Theorem 5), functions u satisfying the boundary and compatibility conditions belonged to the linear space S_0. Each approximating solution u_m then automatically satisfied the boundary and compatibility conditions, and it was possible to obtain convergence to a classical solution in suitable Banach spaces. Functions u_0 satisfying (5.12), however, do not form a linear space, and the method of Theorem 5 is inapplicable.

To overcome this problem we first consider an associated linear equation for which it is possible to obtain a classical solution using the Galerkin method. The following theorem is a statement of this result.

THEOREM 10. *Let f be a continuous real valued function on* $[0, T]$ *such that*

$$f, \dot{f}, \ddot{f} \in L^\infty(0, T). \tag{5.13}$$

Let $u_0 \in H^6(\Omega)$ with

$$u_0 = u_0^{(1)} = \alpha u_0^{(4)} - f(0)\, u_0^{(2)} = \alpha u_0^{(5)} - f(0)\, u_0^{(3)} = 0 \qquad \text{at} \qquad x = 0, l, \tag{5.14}$$

where $\alpha > 0$. Let

$$u_1 \in X = H_0^2(\Omega) \cap H^4(\Omega).$$

Then there exists a unique function $y \equiv y(x, t)$ with

$$\begin{aligned} &y \in L^\infty(0, T; H_0^2(\Omega) \cap H^6(\Omega)),\\ &\dot{y} \in L^\infty(0, T; X),\\ &\ddot{y} \in L^\infty(0, T; H_0^2(\Omega)),\\ &\dddot{y} \in L^\infty(0, T; L^2(\Omega)),\\ &y \in C^1(\bar{Q}) \cap [C^5(\bar{\Omega}) \times C^2([0, T])], \end{aligned} \tag{5.15}$$

such that y satisfies the linear equation

$$\ddot{y} + \alpha y^{(4)} - f(t)\, y^{(2)} = 0 \tag{5.16}$$

and the initial conditions

$$y(0) = u_0\,, \qquad \dot{y}(0) = u_1\,. \tag{5.17}$$

Remarks. (i) The conditions (5.14) are well defined by the embedding theorems.

(ii) (5.16) is the equation for the deflection of a beam with time varying axial force proportional to $f(t)$. Theorem 10 relates the smoothness of the deflection to the smoothness of f.

The proof of Theorem 10 needs several preliminary results, which are given in the next subsection.

(iv) *A Special Basis for the Galerkin Method*

LEMMA 9. *Let*

$$\gamma = f(0)/\alpha.$$

Consider the ordinary differential equation $Lw = \lambda w$ subject to the boundary conditions $w = w^{(1)} = 0$ at $x = 0$ and $x = l$, where

$$Lw \equiv w^{(4)} - \gamma w^{(2)}.$$

Then

(i) *There exist an infinity of eigenvalues λ_i whose absolute values are unbounded and for which zero is not an accumulation point.*

(ii) *To each eigenvalue λ_i corresponds a unique normalized eigenfunction w_i. For convenience, enumerate the λ_i so that $0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$. Zero $(= \lambda_0)$ may be an eigenvalue, in which case let w_0 be the corresponding non-trivial eigenfunction.*

(iii) *The normalized eigenfunctions w_i form a basis of $L^2(\Omega)$. Any $g \in L^2(\Omega)$ can be expanded in a series $g(x) = \sum_i (w_i, g)\, w_i(x)$, convergence holding in $L^2(\Omega)$.*

Proof. The lemma is a well known consequence of the theory of Green's functions and compact operators. See Coddington and Levinson [8, Chapter 7], Courant and Hilbert [9], and Everitt [14]. The uniqueness of w_i is easy to prove but unnecessary for our purposes. □

LEMMA 10. *Let M be the subspace of $L^2(\Omega)$ generated by w_0 if $\lambda = 0$ is an* *eigenvalue of L, and be empty otherwise. Let $M^\perp$ be the orthogonal complement of M in $L^2(\Omega)$. Then for all $y \in M^\perp \cap X$,*

$$| y | \leqslant | \lambda_1 |^{-1} | Ly | .$$

Proof. From Lemma 9, $y = \sum_{r=1}^{\infty} a_r w_r$ in $L^2(\Omega)$, where $a_r = (y, w_r)$. Since $Ly \in M^\perp$, $Ly = \sum_{r=1}^{\infty} b_r w_r$ in $L^2(\Omega)$, where $b_r = (Ly, w_r) = \lambda_r a_r$. By Parseval's relation,

$$| Ly |^2 = \sum_{r=1}^{\infty} \lambda_r^2 a_r^2, \qquad | y |^2 = \sum_{r=1}^{\infty} a_r^2.$$

The result follows. □

LEMMA 11. *For all $y \in M^\perp \cap X$, $| y^{(4)} | \leqslant C | Ly |$.*

Proof.

$$\begin{aligned} | y^{(2)} |^2 = (y^{(4)}, y) &= (y^{(4)} - \gamma y^{(2)}, y) + \gamma(y^{(2)}, y) \\ &\leqslant | y^{(4)} - \gamma y^{(2)} | \, | y | + | \gamma | \, | y^{(2)} | \, | \lambda_1 |^{-1} | y^{(4)} - \gamma y^{(2)} | \\ &\leqslant C | y^{(2)} | \, | y^{(4)} - \gamma y^{(2)} | . \end{aligned}$$

Hence

$$| y^{(2)} | \leqslant C | y^{(4)} - \gamma y^{(2)} | .$$

Now

$$\begin{aligned} | y^{(4)} |^2 &= (y^{(4)}, y^{(4)} - \gamma y^{(2)}) + \gamma(y^{(4)}, y^{(2)}) \\ &\leqslant | y^{(4)} | \, | y^{(4)} - \gamma y^{(2)} | + C | y^{(4)} | \, | y^{(4)} - \gamma y^{(2)} | . \end{aligned}$$

Hence

$$| y^{(4)} | \leqslant C | y^{(4)} - \gamma y^{(2)} | . \quad \square$$

Let

$$Y = \{ y \in H^6(\Omega) \mid y, y^{(1)}, y^{(4)} - \gamma y^{(2)}, y^{(5)} - \gamma y^{(3)} \in H_0^1(\Omega) \},$$

which is a closed subspace of $H^6(\Omega)$ and hence is a Hilbert space. The main result of this subsection is

THEOREM 11. *$\{w_j\}$ is a basis of X and of Y.*

Proof. (a) We first prove that $\{w_j\}$ is a basis of X. Let $v \in X$ and suppose $\epsilon > 0$. Then $v = v_0 + v_1$, where $v_0 = 0$ or μw_0 and $v_1 \in M^\perp \cap X$. Since $Lv_1 \in M^\perp$, there exists a finite linear combination Σ of the w_i $(i = 1, 2,...)$ such that $| Lv_1 - \Sigma | < \epsilon$. By replacing w_j by w_j/λ_j we may write $\Sigma = L\Sigma_1$, where Σ_1 is another finite linear combination of the w_i. By Lemmas 10 and 11, $\| v_1 - \Sigma_1 \|_X < C\epsilon$. Hence $\| v - (\Sigma_1 + v_0) \|_X < C\epsilon$.

(b) To prove that $\{w_j\}$ is a basis of Y, first suppose that $\chi \in Y \cap H^8(\Omega)$. Then $\chi = \chi_0 + \chi_1$, where $\chi_0 = 0$ or μw_0 and $\chi_1 \in M^\perp \cap Y \cap H^8(\Omega)$. Given $\epsilon > 0$, there exists a linear combination Σ_2 of the w_i $(i = 1, 2,...)$ such that $|L^2(\chi_1 - \Sigma_2)| < \epsilon$. Since $L(\chi_1 - \Sigma_2) \in M^\perp \cap X$, by Lemma 11 it follows that $|L(\chi_1^{(4)} - \Sigma_2^{(4)})| < C\epsilon$. Using the relations

$$(\chi_1 - \Sigma_2)^{(r+4)} = L(\chi_1^{(r)} - \Sigma_2^{(r)}) + \gamma(\chi_1^{(r+2)} - \Sigma_2^{(r+2)}) \qquad r = 1, 2, 3, 4,$$

it is easy to prove that

$$\| \chi_1 - \Sigma_2 \|_{H^8(\Omega)} < C\epsilon.$$

Thus

$$\| \chi - (\Sigma_2 + \chi_0) \|_{H^8(\Omega)} < C\epsilon,$$

showing that $\{w_j\}$ is a basis of $Y \cap H^8(\Omega)$.

Suppose now that $y \in Y$. There exist $\{y_r\} \in Y \cap H^8(\Omega)$ such that $\| y_r - y \|_Y \to 0$. Given $\epsilon > 0$, choose r such that $\| y_r - y \|_Y < \epsilon/2$ and a linear combination Σ of the w_i such that $\| y_r - \Sigma \|_Y < \epsilon/2$. Then $\| y - \Sigma \|_Y < \epsilon$. Thus $\{w_j\}$ is a basis of Y. □

Note. $\{w_j\}$ is not an orthogonal basis of X or Y.

(v) *Proof of Theorem* 10

Approximating solutions. We use the basis $\{w_j\}$ of X and Y discussed in the last subsection. The approximating solutions y_m are of the form

$$y_m(t) = \sum_{i=1}^{m} h_{im}(t)\, w_i$$

and satisfy in $[0, t_m]$

$$(\ddot{y}_m(t) + \alpha y_m^{(4)}(t) - f(t)\, y_m^{(2)}(t), w_j) = 0 \qquad 1 \leqslant j \leqslant m \tag{5.18}$$

and the initial conditions

$$y_m(0) = y_{m0} \to u_0 \quad \text{in} \quad Y,$$
$$\dot{y}_m(0) = y_{m1} \to u_1 \quad \text{in} \quad X.$$

From the assumptions on f it follows that

$$\dddot{y}_m\,,\ \ddddot{y}_m \in L^\infty(0, T; Y). \tag{5.19}$$

Estimates. Since

$$\tfrac{1}{2}\,(d/dt)\,(|\dot{y}_m|^2 + \alpha\, |y_m^{(2)}|^2) = f(t)\,(y_m^{(2)}, \dot{y}_m) \leqslant |f(t)|\, |y_m^{(2)}|\, |\dot{y}_m|,$$

from Lemma 3 follow the energy bounds

$$|y_m^{(2)}|, |\dot{y}_m| < C. \tag{5.20}$$

Differentiate (5.18) with respect to t to obtain

$$(\dddot{y}_m(t) + \alpha\dot{y}_m^{(4)}(t) - f(t)\,\dot{y}_m^{(2)}(t) - \dot{f}(t)\,y_m^{(2)}(t), w_j) = 0. \tag{5.21}$$

The bounds

$$|\ddot{y}_m|, |\dot{y}_m^{(2)}| < C \tag{5.22}$$

follow in the same way as (5.10).

Next differentiate (5.21) with respect to t to obtain

$$(\ddddot{y}_m(t) + \alpha\ddot{y}_m^{(4)}(t) - f(t)\,\ddot{y}_m^{(2)}(t) - 2\dot{f}(t)\,\dot{y}_m^{(2)}(t) - \ddot{f}(t)\,y_m^{(2)}(t), w_j) = 0. \tag{5.23}$$

Since $\dddot{y}_m$ and $\ddddot{y}_m \in L^\infty(0, T; Y)$ it follows from Lemma 5 that $|\dddot{y}_m(t)|^2$ and $|\ddot{y}_m^{(2)}(t)|^2$ are absolutely continuous functions of t with derivatives $2(\dddot{y}_m(t), \ddddot{y}_m(t))$ and $2(\ddot{y}_m^{(2)}(t), \dddot{y}_m^{(2)}(t))$ a.e.. Multiplying (5.23) by $\dddot{h}_{jm}$ and summing for $j = 1, \ldots, m$ thus gives

$$\begin{aligned} &\tfrac{1}{2}(d/dt)(|\dddot{y}_m|^2 + \alpha|\ddot{y}_m^{(2)}|^2) \\ &\quad = (f(t)\,\ddot{y}_m^{(2)} + 2\dot{f}(t)\,\dot{y}_m^{(2)} + \ddot{f}(t)\,y_m^{(2)}, \dddot{y}_m) \\ &\quad \leqslant |\dddot{y}_m|\,(|f(t)|\,|\ddot{y}_m^{(2)}| + 2\,|\dot{f}(t)|\,|\dot{y}_m^{(2)}| + |\ddot{f}(t)|\,|y_m^{(2)}|) \\ &\quad \leqslant C(1 + |\dddot{y}_m|^2 + \alpha|\ddot{y}_m^{(2)}|^2). \end{aligned}$$

Hence if we can show that $|\dddot{y}_m(0)|$ and $|\ddot{y}_m^{(2)}(0)|$ are uniformly bounded, it will follow that

$$|\dddot{y}_m|, |\ddot{y}_m^{(2)}| < C. \tag{5.24}$$

But from (5.21),

$$|\dddot{y}_m(0)|^2 = |(\alpha y_{m1}^{(4)} - f(0)\,y_{m1}^{(2)} - \dot{f}(0)\,y_{m0}^{(2)}, \dddot{y}_m(0))|$$

and consequently

$$|\dddot{y}_m(0)| < C.$$

To show that $|\ddot{y}_m^{(2)}(0)| < C$ we use the properties of the basis $\{w_j\}$. From (5.18) we deduce that

$$(\ddot{y}_m(0) + \alpha y_{m0}^{(4)} - f(0)\,y_{m0}^{(2)}, \ddot{y}_m(0)) = 0 \tag{5.25}$$

and that

$$(\ddot{y}_m(0) + \alpha y_{m0}^{(4)} - f(0)\, y_{m0}^{(2)}, y_{m0}^{(4)} - \gamma y_{m0}^{(2)}) = 0. \tag{5.26}$$

Add (5.25) to $\alpha \times$ (5.26) and use the fact that $f(0) = \alpha\gamma$. Then

$$|\ddot{y}_m(0) + \alpha y_{m0}^{(4)} - f(0)\, y_{m0}^{(2)}|^2 = 0.$$

Hence

$$\ddot{y}_m(0) + \alpha y_{m0}^{(4)} - f(0)\, y_{m0}^{(2)} = 0$$

and $|\ddot{y}_m^{(2)}(0)| < C$ follows from the assumptions on y_{m0}.

Convergence. We can now extract a subsequence $\{y_\mu\}$ of $\{y_m\}$ satisfying

$$\begin{aligned} y_\mu &\to y && \text{in} && L^\infty(0, T; H_0^2(\Omega)) \text{ weak*},\\ \dot{y}_\mu &\to \dot{y} && \text{in} && L^\infty(0, T; H_0^2(\Omega)) \text{ weak*},\\ \ddot{y}_\mu &\to \ddot{y} && \text{in} && L^\infty(0, T; H_0^2(\Omega)) \text{ weak*},\\ \dddot{y}_\mu &\to \dddot{y} && \text{in} && L^\infty(0, T; L^2(\Omega)) \text{ weak*},\\ \text{and} \quad y_\mu &\to y && \text{in} && H^1(Q) \text{ strongly and a.e.} \end{aligned}$$

Hence

$$(\ddot{y}, v) + \alpha(y^{(2)}, v^{(2)}) = f(t)\,(y^{(2)}, v) \qquad \text{for all } v \in X. \tag{5.27}$$

By the same method as in Theorem 9, $y \in L^\infty(0, T; X)$ and

$$\ddot{y} + \alpha y^{(4)} - f(t)\, y^{(2)} = 0 \qquad \text{a.e. in} \qquad [0, T]. \tag{5.28}$$

Differentiating (5.28) once and twice with respect to x shows that $y^{(5)}, y^{(6)} \in L^\infty(0, T; L^2(\Omega))$, and hence that y satisfies (5.15). That y satisfies (5.17) follows in the usual manner. □

(vi) *Classical Solutions*

The existence of a classical solution to (3.1), (3.2) and (3.4) now follows rapidly from Theorem 10.

THEOREM 12. *Let $u_0 \in H^6(\Omega)$ and satisfy* (5.12). *Let $u_1 \in X$. Then the unique solution u in Theorem 9 is such that*

$$\begin{aligned} & u \in L^\infty(0, T; H_0^2(\Omega) \cap H^6(\Omega)),\\ & \dot{u} \in L^\infty(0, T; X),\\ & \ddot{u} \in L^\infty(0, T; H_0^2(\Omega)),\\ & \dddot{u} \in L^\infty(0, T; L^2(\Omega)),\\ \textit{and} \quad & u \in C^1(\bar{Q}) \cap [C^5(\bar{\Omega}) \times C^2([0, T])]. \end{aligned} \tag{5.29}$$

Proof. Let

$$f(t) = \beta + k \mid u^{(1)}(t)\mid^2.$$

Then

$$\dot{f}(t) = 2k(u^{(1)}(t), \dot{u}^{(1)}(t)) = -2k(u^{(2)}(t), \dot{u}(t))$$

and

$$\ddot{f}(t) = -2k(\dot{u}^{(2)}(t), \dot{u}(t)) - 2k(u^{(2)}(t), \ddot{u}(t)).$$

Hence $f, \dot{f}, \ddot{f} \in L^\infty(0, T)$. Also

$$f(0) = \beta + k \mid u_0^{(1)} \mid^2.$$

Theorem 10 now guarantees the existence of y satisfying (5.15)–(5.17). Subtract (5.16) from (5.7), letting $w = u - y$. Thus

$$\ddot{w} + \alpha w^{(4)} - f(t)\, w^{(2)} = 0. \tag{5.30}$$

Hence

$$(\ddot{w}, \dot{w}) + \alpha(w^{(4)}, \dot{w}) = f(t)\,(\dot{w}, w^{(2)})$$

and so

$$\frac{d}{dt}\left(\frac{1}{2} \mid \dot{w} \mid^2 + \frac{\alpha}{2} \mid w^{(2)} \mid^2\right) \leqslant C \mid \dot{w} \mid \mid w^{(2)} \mid .$$

Thus $w = 0$ and $u = y$. The theorem follows. □

Remark. Clearly u satisfies the boundary conditions (3.4) and the compatibility conditions

$$\alpha u^{(4)} - (\beta + k \mid u^{(1)} \mid^2)\, u^{(2)} = \alpha u^{(5)} - (\beta + k \mid u^{(1)} \mid^2)\, u^{(3)} = 0$$

at $x = 0$ and l.

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